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THE ARITHMETIC GENUS OF AN ALGEBRAIC MANIFOLD IMMERSED IN ANOTHER.

BY S. LEFSCHETZ.

Introduction.

1. Let V_r be an r -dimensional irreducible algebraic variety¹ immersed in an $(r + k)$ -fold space S_{r+k} , in which it has no singularities, or else is the projection of a non-singular variety in a higher space. There can be found² in V_r a fixed number ρ of *hypersurfaces*, $C_1, C_2, \dots, C_\rho$, such that to any other C , there corresponds a relation of *equivalence* in the sense of Severi, of the type

$$\lambda C = \lambda_1 C_1 + \lambda_2 C_2 + \dots + \lambda_\rho C_\rho,$$

where the λ 's are integers, while no such relation exists between $C_1, C_2, \dots, C_\rho$. These last hypersurfaces form a *base* in V_r , and the λ 's may be called the *base characters* of C . The object of this paper is *to find the arithmetic genus of a manifold complete intersection in V_r , in terms of the base characters of the hypersurfaces which determine it.* The case of $r = 1, k = 2$ was first derived by Salmon,³ that of $r = 1, k = n - 1$ by Veronese,⁴ that of $r = 2, k = n - 2$ by Severi.⁵ For an algebraic surface the problem has already been solved by Severi,⁶ but as his method admits of no easy generalization, the one followed here is completely different, and consists in the extensive use of a symbolism, already somewhat employed by the Italian geometer.⁷

In part I, some auxiliary properties are considered, notably the extension of certain formulas to virtual manifolds. In part II the solution of the fundamental problem is given, while in part III are found applications to loci of flats and to complete intersections in an S_r .

¹ A d -flat will be designated throughout by S_d . The name *variety* will be reserved for V_r . Any of its subvarieties will be called a *manifold*. If the manifold has $r - 1$ dimensions, it will be called a *hypersurface*.

² Severi, *La base minima*. . . . Ann. Ec. Norm. Sup. (1908), p. 467.

³ Geometry of three dimensions. 4th edition (1882), p. 309-311.

⁴ Über die Methode des Projicirens und Schneidens. Math. Ann., vol. 19 (1882), p. 203.

⁵ Su alcune questioni di postulazione. Rendic. del Circ. Mat. di Palermo, vol. 17 (1903), p. 76-79.

⁶ Sulla base per la totalità delle curve. . . . Math. Ann. (1906), p. 221.

⁷ Fundamenti per la geometria sulle varietà algebriche. Rendic. del Circolo Mat. di Palermo (1909), p. 48, footnote.

The assumption is made throughout that all manifolds considered have no assigned singularities.

2. Let $v(l)$ be the *postulation* of V_r , that is the number of conditions imposed upon a V_{r+k-1} , of order l sufficiently high, when it goes through V_r . According to Hilbert⁸

$$v(l) = k_0 \binom{l+r}{r} + k_1 \binom{l+r-1}{r-1} + \cdots + k_{r-1}(l+1) + k_r,$$

the k 's being certain constants. The *arithmetic genus*,⁹ or simply *genus*, of V_r , which will be designated by $[V_r]$, when $r > 0$, is given by

$$[V_r] = (-1)^r(k_0 + k_1 + \cdots + k_r - 1) = (-1)^r(v(0) - 1).$$

When $r = 0$, V_r is a group of points, and the genus $[V_r]$, by definition, the number of points in the group.

The letters $A, B, C, \dots$ will be used to designate hypersurfaces, and in accordance with the notion of equivalence already referred to, two hypersurfaces belonging to the same continuous algebraic system will be designated by the same letter. Let $A, B, \dots, C$, be a certain number h of hypersurfaces. If their complete intersection has exactly $r - h$ dimensions, it will be designated by $AB \cdots C$.¹⁰ When no confusion may arise, we will sometime designate it by M_{r-h} , or M , and write $M = AB \cdots C$. What is meant by a hypersurface being the sum of two others is well known, as well as the notation to indicate it. This will be extended here, and we will denote the intersection of M with the sum of B and C by $M(B + C)$. The genus of $M = AB \cdots C$ will be denoted by $[M]$, or by $[AB \cdots C]$.

In addition to these notations, the following will be introduced: Let $F(A, B, \dots, C)$ be a polynomial, or a series proceeding according to positive integral powers of $A, B, \dots, C$. Then

(a) By $[F(A, B, \dots, C)]$ we will denote the result obtained when the constant term is left unchanged, and when in place of the term in $A^a B^b \cdots C^c$ we put

$$[A^a B^b \cdots C^c], \text{ if its degree is inferior to } r;$$

zero, if its degree exceeds r .

(b) By $\{F(A, B, \dots, C)\}$ we will designate the same result as the pre-

⁸ Über die Theorie der algebraischen Formen. Math. Ann., vol. 36 (1890), p. 512, 520.

⁹ Severi. Fundamenti. . . . p. 41. The definition given here differs from his only when $r = 0$.

¹⁰ In § 5 a meaning will be attributed to $AB \cdots C$, even when this complete intersection is of less than $r - h$ dimensions. In this case, however, there is no connection between the actual intersection and the symbol $AB \cdots C$.

ceding, as far as terms of degree less than r are concerned, but in place of $A^a B^b \dots C^c$ we will set

$$[A^a B^b \dots C^c] - 1, \text{ if the degree is } r,$$

$$(-1)^{k-1}, \text{ if the degree is } r + k, k > 0.$$

In particular if $M = A^a B^b \dots C^c$, with $h = a + b + \dots + c$, then when $h \leq r$, $[M]$, $\{M\}$, are the genus as defined above, or as defined by Severi, respectively, while when $h > r$, $[M] = 0$, $\{M\} = (-1)^{r-h-1}$.

From the above follows

$$\{F(A, B, \dots, C)\} = [F(A, B, \dots, C)] + N,$$

where N is a certain constant. To obtain its value, remark that the term $\alpha \cdot A^a B^b \dots C^c$ will contribute to N an amount

$$(-1)^{k-1} \cdot \alpha, \text{ if its degree is } r + k, k \geq 0;$$

$$\text{zero, if its degree is less than } r.$$

Hence, if $F(x) = \sum \alpha_n x^n$ is the result obtained when in $F(A, B, \dots, C)$ we make $A = B = \dots = C = x$, then

$$(-1)^{r-1} N = \sum_{n=r}^{\infty} (-1)^n \alpha_n = F(-1) - \sum_{n=0}^{r-1} (-1)^n \alpha_n.$$

This result will be useful later.

I. Preliminary formulas. Virtual manifolds.

3. Let¹¹ A, C_1, C_2 , be hypersurfaces in V_r , such that $A = C_1 + C_2$. Then whatever r may be

$$\{A\} = \{(1 + C_1)(1 + C_2) - 1\}. \quad (1)$$

For if $r > 1$, this is a formula of Severi's,¹² since $\{M\}$ is the genus of M according to his definition in that case. If $r = 1$, $\{A\} = [A] - 1$, $\{C_1\} = [C_1] - 1$, $\{C_2\} = [C_2] - 1$, $\{C_1 C_2\} = +1$, and the formula reduces to

$$[A] = [C_1 + C_2],$$

which is true, since A, C_1, C_2 , are groups of points, and their genus, the number of points in each group.

Let $M = B_1 \dots B_h$ be a complete intersection in V_r , $h \leq r - 1$. If we apply (1) in M , we get

$$\{MA\} = \{M((1 + C_1)(1 + C_2) - 1)\}. \quad (2)$$

¹¹ Restrictions upon the formulas of this and the next paragraph, will be considered in § 5.

¹² Fundamenti. . . , p. 42.

If $h = r + k$, $k \geq 0$, the quantities involved are mere symbols. From § 2 follows, however, $\{MA\} = \{MC_1\} = \{MC_2\} = -\{MC_1C_2\} = (-1)^k$ and the relation is still correct. Hence, if $F(A, B, \dots, C)$ is a polynomial or a power series

$$\{F(A, B, \dots, C)\} = \{F((1 + C_1)(1 + C_2) - 1, B, \dots, C)\}. \quad (3)$$

From this follows easily the generalization of (1). I say that if $A = \sum_{i=1}^n C_i$, then

$$\{A\} = \left\{ \prod_{i=1}^n (1 + C_i) - 1 \right\}. \quad (4)$$

For, first this is true if $n = 2$. Grant that it holds when A is the sum of less than n hypersurfaces. Then, if $C' = C_{n-1} + C_n$,

$$\{A\} = \left\{ (1 + C') \prod_{i=1}^{n-2} (1 + C_i) - 1 \right\} = \left\{ (1 + C_{n-1})(1 + C_n) \prod_{i=1}^{n-2} (1 + C_i) - 1 \right\}$$

which is the formula to be proved.

From (4) follows the generalization of (3), thus: If A designates the same sum as in (4),

$$\{F(A, B, \dots, C)\} = \left\{ F\left(\prod_{i=1}^n (1 + C_i) - 1, B, \dots, C \right) \right\} \quad (5)$$

which is derived from (4), like (3) from (1).

Finally if $A_i = \sum_{k=1}^{n_i} C_{ik}$, ($i = 1, 2, \dots, h$), applying (5) we obtain

$$\{A_1 A_2 \dots A_h\} = \left\{ \prod_{i=1}^h \left(\prod_{k=1}^{n_i} (1 + C_{ik}) - 1 \right) \right\} \quad (6)$$

4. From § 2 and formula (4) follows, that if $A = \sum_{i=1}^n C_i$,

$$\{A\} = \left\{ \prod_{i=1}^n (1 + C_i) - 1 \right\} = \left[\prod_{i=1}^n (1 + C_i) - 1 \right] + N$$

where, in this case, N is a certain integer, and from what has been shown in that paragraph,

$$(-1)^{r-1}N = ((1+x)^n - 1)_{x=-1} - \sum_{k=1}^{r-1} (-1)^k \binom{n}{k};$$

$$\therefore N = (-1)^r \sum_{k=0}^{r-1} (-1)^k \binom{n}{k} = -\binom{n-1}{r-1};$$

$$\therefore \{A\} = \left[\prod_{i=1}^n (1 + C_i) - 1 \right] - \binom{n-1}{r-1}, \quad (7)$$

formulas which will be of frequent use.¹³

¹³ For the simplification involved, see Netto, Lehrbuch der Kombinatorik, p. 248, formula 14.

We have next, from (6), and with the same notations,

$$\{A_1 A_2 \cdots A_h\} = \left[\prod_{i=1}^h \left(\prod_{k=1}^{n_i} (1 + C_{ik}) - 1 \right) \right] + N',$$

where N' is again an integer. This integer remains the same if all the C 's coincide with the same hypersurface C . Let M be what $A_1 A_2 \cdots A_h$ becomes in that case, and $n = \sum n_i$. We have

$$\begin{aligned} \{M\} &= \left\{ \prod_{i=1}^h [(1 + C)^{n_i} - 1] \right\} \\ &= \{(1 + C)^n - \sum_i (1 + C)^{n-n_i} + \sum_{i,j, i \neq j} (1 + C)^{n-n_i-n_j} + \cdots + (-1)^h\}. \end{aligned}$$

From (4) and (7) follows

$$1 + \{\overline{kC}\} = \{(1 + C)^k\} = [(1 + C)^k] - \binom{k-1}{r-1};$$

$$\begin{aligned} \therefore \{M\} &= [(1 + C)^n - \sum_i (1 + C)^{n-n_i} + \sum_{i,j, i \neq j} (1 + C)^{n-n_i-n_j} + \cdots + (-1)^h] \\ &\quad - \left(\binom{n-1}{r-1} - \sum_i \binom{n-n_i-1}{r-1} + \sum_{i,j, i \neq j} \binom{n-n_i-n_j-1}{r-1} \cdots \right. \\ &\quad \left. + (-1)^{h-1} \sum_i \binom{n_i-1}{r-1} \right). \end{aligned}$$

The last parenthesis is equal to $-N'$.

$$\begin{aligned} \therefore \{A_1 A_2 \cdots A_h\} &= \left\{ \prod_{i=1}^h \left(\prod_{k=1}^{n_i} (1 + C_{ik}) - 1 \right) \right\} - \binom{n-1}{r-1} \\ &\quad + \sum_i \binom{n-n_i-1}{r-1} - \sum_{i,j, i \neq j} \binom{n-n_i-n_j-1}{r-1} \quad (8) \\ &\quad + \cdots + (-1)^h \sum_i \binom{n_i-1}{r-1}. \end{aligned}$$

Formulas (7) and (8) are more convenient for applications than those of § 3.

5. Virtual Manifolds.—The formulas developed so far do not always hold. Thus (2), which can be considered as the foundation of the rest, can be applied only when the dimensionality $r - h$ of M is > 1 , if its intersection with $C_1 C_2$ has $r - h - 2$ dimensions. It is the purpose of what follows to remove these restrictions. This will be done by the introduction, with Severi, of *virtual manifolds*.

If A, B, C are effective, and $A = B + C$, C is called the difference of A and B , and we write $C = A - B$. It is well known that $C = (A + D) - (B + D)$ for any effective D . Given that A, B are effective; if there

is no effective C such that $A = B + C$, we say that C is *virtual*. Similarly, if there is no effective $r - h$ dimensional manifold $M_{r-h} = C_1 C_2 \cdots C_h$, we will say that M_{r-h} is virtual. This may happen only if the C 's, though all effective, have in common a manifold of less than $r - h$ dimensions, or if some of the C 's themselves are virtual. When M_{r-h} is virtual, it is a mere symbol; by definition it is unchanged when the C 's are arbitrarily permuted.

6. Let A, B, C be effective and such that $C = A - B$, and also that BC^k be effective for $k \leq r - 1$. By repeatedly applying (1) and (2), we obtain

$$\begin{aligned}\{A\} &= \{B + C + BC\}, \\ \{BA\} &= \{B^2 + BC + B^2C\}, \\ \{B^{r-1}A\} &= \{B^r + B^{r-1}C + B^rC\}.\end{aligned}$$

Hence, by eliminating $BC, B^2C, \dots, B^{r-1}C$,

$$\{C\} = \{(A - B)(1 - B + B^2 - \dots + (-1)^{r-1}B^{r-1}) + (-1)^r\{B^rC\};$$

$$\therefore \{C\} = \left[\frac{A - B}{1 + B} \right] + (-1)^r, \quad (9)$$

since $\{B^rC\} = +1$, (§ 2). In this formula and those to be developed, the fractions stand for their expansion in powers of $A, B, \dots$. Formula (9) may be considered as the "inversion" of (1). Similarly if the manifolds

$$M = C_1 C_2 \cdots C_h, \quad MCB^k, \quad k \leq r - h - 1,$$

are all effective, the equation

$$\{MC\} = \left[M \frac{A - B}{1 + B} \right] + (-1)^{r-h} \quad (10)$$

gives the inversion of (2). If $C_i = A_i - B_i$, ($i = 1, 2, \dots, h$), and if the manifolds

$$\begin{aligned}A_{i_1} A_{i_2} \cdots A_{i_s} B_1^{b_1} B_2^{b_2} \cdots B_h^{b_h}, \quad (k + \sum b_i < r), \\ C_1 C_2 \cdots C_s B_1^{b_1} B_2^{b_2} \cdots B_h^{b_h}, \quad (s \leq h, s + \sum b_i < r),\end{aligned}$$

are all effective, the inversion of (6), obtained by repeated application of (10), is given by

$$\{C_1 C_2 \cdots C_h\} = \left[\prod_{i=1}^h \frac{(A_i - B_i)}{1 + B_i} \right] + (-1)^{r+1-h}. \quad (11)$$

Suppose now that $M = C_1 C_2 \cdots C_h$ be virtual. We can always find effective hypersurfaces A_i, B_i , such that $C_i = A_i - B_i$, then by definition

(a) $\{M\}$ is given by (11) as if M were effective.

(b) The relations between $\{M\}$ and $[M]$ are the same as when the manifold is effective.

If all manifolds $A_1, A_2, \dots, A_i, B_1^{b_1} B_2^{b_2} \dots B_h^{b_h}$, $k + \sum b_i < r$, are effective, this definition is complete. If not $[M]$ will be determined in terms of genera of manifolds, not all effective perhaps, but all situated in effective hypersurfaces, that is in a V_{r-1} , and the definition is then recurrent. In all cases a repeated application of (11) will lead by a finite number of operations to effective genera.

7. The genus of a manifold in the extended field composed of all effective and virtual manifolds has the following fundamental properties which are known to hold in the restricted field, and which justify the definitions just adopted.

I. If $C = A - B$, where A and B are effective, then

$$\{M \cdot C\} = \left[M \cdot \frac{A - B}{1 + B} \right] + (-1)^{r-h}.$$

This follows at once from (11) as applied to $C_1 C_2 \dots C_h C$.

II. If $C = C' + C''$, then

$$\{MC\} = \{M((1 + C')(1 + C'') - 1)\}.$$

Let $C' = A' - B'$, $C'' = A'' - B''$, $C = \overline{A' + A'' - B' + B''}$, where A', B', A'', B'' , are effective. The property to be proved holds for a V_1 , for which, by definition, $[\overline{A - B}] = [A - B]$, so that the verification is immediate. Assume that it be true for any V_{r-1} , we say that it is also true for V_r . We have, by repeated application of property I,

$$\{M(C' + C'' + C'C'' - C)\} = \left[(C' + C'' + C'C'' - C) \prod_{i=1}^h \frac{A_i - B_i}{1 + B_i} \right].$$

If we expand the right-hand side, we obtain a sum of terms such as

$$[A_1 \cdot (C' + C'' + C'C'' - C)M'] - [B_1 \cdot (C' + C'' + C'C'' - C)M'].$$

As A_1 and B_1 are effective hypersurfaces, that is $r - 1$ dimensional varieties, it follows from our assumptions that the expression just written is equal to zero.

$$\therefore \{M((1 + C')(1 + C'') - C - 1)\} = 0,$$

which is what we wished to prove.

Property II means that formula (2), and all formulas derived so far hold whether the manifolds involved are effective or not. As all the relations to be derived are based upon these, no restrictions in regard to "effectiveness" will be necessary.

III. If in $C_i = A_i - B_i$, A_i and B_i be replaced by $A_i + D_i$, $B_i + D_i$, respectively, $[M]$ does not change. In other words, $[M]$ is a function of the differences $A_i - B_i$. We have, by property II,

$$\begin{aligned} (-1)^{r+1-h} + \left[\prod_{i=1}^h \frac{\overline{A_i + D_i} - \overline{B_i + D_i}}{1 + \overline{B_i + D_i}} \right] \\ = \left[\prod_{i=1}^h \frac{(1 + A_i)(1 + D_i) - (1 + B_i)(1 + D_i)}{(1 + B_i)(1 + D_i)} \right] + (-1)^{r+1-h} \\ = \left[\prod_{i=1}^h \frac{A_i - B_i}{1 + B_i} \right] + (-1)^{r+1-h} = \{M\} \end{aligned}$$

as was to be proved.

II. Solution of the fundamental problem.

8. In the solution of the fundamental problem, it is desirable to obtain formulas which hold, whatever the signs of the λ 's as defined in § 1. The relations derived so far are not suitable for this purpose. We propose to show first the existence of solutions of the required nature. This having been done, the solutions will be obtained for a special case, and the generalization will follow easily.

Let $A = C_1 + C_2$. Whatever the positive integer λ , we have

$$\lambda A = \lambda C_1 + \lambda C_2.$$

Applying (7) to both sides of this relation, we obtain

$$[(1 + A)^\lambda - 1] - \binom{\lambda}{r} \frac{1}{r} = [(1 + C_1)^\lambda (1 + C_2)^\lambda - 1] - \binom{2\lambda}{r} \frac{1}{r}.$$

This equation of degree r in λ , has an infinity of roots, hence it is identically true. If we set

$$F(x) = \sum_{k=1}^r (-1)^{k-1} \frac{x^k}{k} + (-1)^{r-1} \sum_{k=1}^{r-1} \frac{1}{k},$$

we obtain by equating terms of the first degree in λ ,

$$[F_r(A)] = [F_r(C_1)] + [F_r(C_2)].$$

This also shows that if $C_2 = A - C_1$, then

$$[F_r(C_2)] = [F_r(A)] - [F_r(C_1)].$$

Hence, if $\lambda C = \sum_{i=1}^p \lambda_i C_i$, then

$$\lambda [F_r(C)] = \left[\sum_{i=1}^p \lambda_i F_r(C_i) \right]. \quad (12)$$

Let $M = D_1 D_2 \cdots D_h$. If we reason from (2) as we just have from (1), and set

$$r - h = r'; \quad F_{r'}(MC) = \sum_{k=1}^{r'} (-1)^{k-1} \frac{MC^k}{k} + (-1)^{r'-1} \sum_{k=1}^{r'-1} \frac{1}{k}$$

we can show that

$$\lambda[F_{r'}(MC)] = \left[\sum_{i=1}^p \lambda_i F_{r'}(MC_i) \right]. \quad (13)$$

Formulas (12) and (13) hold whatever the signs of the λ 's.

In the case $r = 2$, $[F_2(C)] = \frac{1}{2}[2C - C^2 - 2]$. The double of this last quantity has already been considered by Severi,¹⁴ who proved (12) for $r = 2$, by considering the intersection of C with the canonic curves of V_2 , and made this formula the central point in his solution of the fundamental problem for $r = 2$. But even before this, Castelnuovo and Enriques had shown that if $[F_2(C)] < 0$, for any continuous system of curves on the surface, the latter was birationally transformable into a scroll.

9. Let $(C_1, C_2, \dots, C_p)$ form a base on V_r . For any C not belonging to the base, we have $\lambda C = \sum_{i=1}^p \lambda_i C_i$. Then

THEOREM. *The genus $[C_i^h C^k]$ is a linear function of the genera $[C_1^{h_1} C_2^{h_2} \cdots C_p^{h_p}]$, with coefficients equal to polynomials in the ratios λ_i/λ , the coefficients of these polynomials being independent of the signs of the λ 's.*

The proof will be by induction. The proposition is true: (a) If $h + k > r$, for then $[C_i^h C^k] = 0$. (b) If $h + k = s$, $k = 0$, that is for $[C_i^h]$. Now apply (13), making $M = C_i^{r-1}$. It gives

$$\lambda[CC_i^{r-1}] = \sum_{j=1}^p \lambda_j [C_j C_i^{r-1}],$$

which holds whatever the signs of the λ 's, and shows that the theorem is true for $[CC_i^{r-1}]$, that is when $h + k = r$, $k = 1$. To prove it in the general case, assume that it holds if $h + k > s$, or if $h + k = s$, $k < t$. We say that it also holds when $h + k = s$, $k = t$. To show this, apply (10) with $M = C_i^{s-t} C^{t-1}$. It gives

$$\lambda \sum_{k=1}^{r-s+1} (-1)^{k-1} [C_i^{s-t} C^{t-1+k}] + (-1)^{r-s} \lambda \sum_{k=1}^{r-s} \frac{1}{k} = \sum_{j=1}^p \lambda_j [F_{r-s+1}(C_i^{s-t} C^{t-1} \cdot C_j)],$$

which also holds whatever the signs of the λ 's. This relation, if solved for $[C_i^{s-t} C^t]$, shows that this genus is a linear combination of others of the type $[C_i^h C^k]$, such that $h + k > s$, or $h + k = s$, $k < t$, with coef-

¹⁴ Sulla base per la totalità delle curve, l. c., p. 224.

ficients of the required character. Since for these last genera the theorem is true by assumption, it is also true for $[C_i^{s-t}C^t]$, which was to be proved.

In the same manner we may prove this more general theorem:

THEOREM. *If D_j is such that $\lambda_j D_j = \sum_i \lambda_i^j C_i$, then $[D_1 D_2 \cdots D_h]$ is a linear combination of the genera $[C_1^{i_1} C_2^{i_2} \cdots C_p^{i_p}]$, with coefficients polynomials in the ratios λ_i^j / λ_j , the coefficients of these polynomials being independent of the signs of the λ 's.*

The two theorems just considered establish the existence of solutions of the fundamental problem, satisfying the condition to hold whatever the signs of the λ 's.

Remark.—The question may be raised, whether the two preceding theorems hold when "division" is not unique,¹⁵ that is when it is possible that $\lambda C = \lambda D$, while $C \neq D$ (λ integer > 1). This question must be answered in the affirmative. This will be proved, if we can show that if $\lambda C = \lambda D$, then $[MC^k] = [MD^k]$, whatever M , for then if $\lambda C = \sum \lambda_i C_i$, $[MC^k]$ will be determined without ambiguity. Let $M = A_1 A_2 \cdots A_h$. If $s + t > r - h$, then

$$[MC^s D^t] = [MC^{s+t}] = [MD^{s+t}] = 0.$$

Suppose that

$$[MC^s D^t] = [MC^{s+t}] = [MD^{s+t}], \quad s + t > s_0,$$

we say that it is still true for $s + t = s_0$. For assume $s + t = s_0$, and apply (13). It follows that

$$\left[\sum_{k=1}^{(r-h+s_0)} (-1)^{k-1} \frac{MC^{s-1+k} D^t}{k} \right] = \left[\sum_{k=1}^{r-h+s_0} \frac{MC^{s-1} D^{t+k}}{k} \right].$$

The only terms which are not equal by assumption are those for which $k = 1$. Hence

$$[MC^s D^t] = [MC^{s-1} D^{t+1}] = [MC^{s-2} D^{t+2}] = \cdots = [MD^s] = [MC^s],$$

as was to be proved.

Definition.—If $[MC^k] = [MD^k]$, whatever the integer k , and the manifold M , the hypersurfaces C, D , are said to be arithmetically equivalent, and we write $C \sim D$. This equivalence has the following properties:

- (a) In its field division is unique, that is if $\lambda C \sim \lambda D$, then $C \sim D$.
- (b) If $C = D$, then $C \sim D$.
- (c) If $C \sim D$, then $\lambda C = \lambda B$, where in general the integer $\lambda > 1$.

For let $|H|$ be the system of hyperplane sections. The property is true on a V_1 ; for two groups of m points on an algebraic curve belong to the

¹⁵ Severi, Ann. Ec. Norm. Sup., 1908, p. 456.

same continuous system. Grant that it holds on a V_{r-1} , we say that it also holds on V_r . We have by assumption $[HMC^k] = [HMD^k]$.

$$\therefore HC \sim HD, \quad \lambda HC = \lambda HD,$$

whatever the hyperplane section H . Hence^{15'} $\lambda C = \lambda D$. In this, we have tacitly assumed that C, D are effective. In the contrary case, we would replace them by $C + kH, D + kH$, the integer k being so chosen that the two new hypersurfaces are effective. From $\lambda(C + kH) = \lambda(D + kH)$, follows then $\lambda C = \lambda D$. The properties here given show the relations between the two kinds of equivalence.

10. We are now ready to solve the fundamental problem. The reasoning used in § 9 could be utilized, this being the method followed by Severi for $r = 2$. It seems however impracticable for $r > 2$, and another will be followed here.

Let us specialize the hypersurface C of § 9, by making $\lambda = 1$, and assuming $\lambda_i \equiv 0; i = 1, 2, \dots, \rho$. Then $C = \sum \lambda_i C_i$, and we can obtain $\{C\}$ by applying (7). C is the sum of $\sum \lambda_i$ hypersurfaces,

$$\therefore \{C\} = \left[\prod_{i=1}^{\rho} (1 + C_i)^{\lambda_i} - 1 \right] - \left(\frac{\sum \lambda_i - 1}{r - 1} \right).$$

This result holds for all positive values of the λ 's. From § 9, follows at once that if $\lambda C = \sum \lambda_i C_i$, and if we set $\mu = (1/\lambda) \sum \lambda_i$, then

$$\{C\} = \left[\prod_{i=1}^{\rho} (1 + C_i)^{\lambda_i/\lambda} - 1 \right] - \left(\frac{\mu - 1}{r - 1} \right). \quad (14)$$

Similarly we may prove the following result:

Consider the hypersurfaces defined by

$$\lambda_j D_j = \sum_i \lambda_i^j C_i \quad (j = 1, 2, \dots, h),$$

and set

$$\mu_j = (1/\lambda_j) \sum_i \lambda_i^j, \quad \mu = \sum \mu_j,$$

then

$$\begin{aligned} \{D_1 D_2 \dots D_h\} &= \left[\prod_{j=1}^h \left(\prod_{i=1}^{\rho} (1 + C_i)^{\lambda_i^j / \lambda_j} - 1 \right) \right] - \left(\frac{\mu - 1}{r - 1} \right) \\ &\quad + \sum_i \left(\frac{\mu - \mu_i - 1}{r - 1} \right) \dots + (-1)^h \sum_i \left(\frac{\mu_i - 1}{r - 1} \right) \end{aligned} \quad (15)$$

and in particular

$$[D_1 D_2 \dots D_r] = \left[\prod_{j=1}^r \left(\frac{1}{\lambda_j} \sum_{i=1}^{\rho} \lambda_i^j C_i \right) \right]. \quad (16)$$

In these formulas, the parentheses raised to fractional exponents, must be

^{15'} Severi, l. c., p. 467.

expanded in series of ascending powers of the C 's, and the result treated in accordance with the definition of the symbol $[\]$. Formula (15) gives the most general solution of the fundamental problem.

Remark.—It is important to remember that these formulas hold whether the manifolds involved are effective or not, and whatever the signs of the λ 's, that is *without any restrictions*. If $D_1 D_2 \cdots D_h$ is virtual, its genus is the quantity defined in § 6. If it is effective, its arithmetic genus is the quantity defined in § 2. In particular (16) gives the number of points common to r hypersurfaces, in terms of their base characters, and is the immediate generalization of a formula of Severi's for $r = 2$.

III. Applications.

11. **Loci of Flats.**—A locus of flats is a V_r locus of $(r - 1)$ -flats forming an irrational pencil. Let p be the genus of the pencil, H an arbitrary hyperplane section of V_r , G one of its generators (generating flats). It may be shown that H, G , form a *minimum base* for the variety.¹⁶ That the latter can be considered as the projection of a non-singular variety from a higher space, will be admitted here, and the results obtained applied to this case. If $k > 1$, the intersection of H^i with G^k has less than $r - i - k$ dimensions.¹⁷ As we have seen in § 5, $H^i G^k$ must then be considered as virtual, and its genus cannot be determined from geometric considerations, but only by considering G as the difference of two appropriate effective hypersurfaces and applying (11). If we set $H_1 = H - G$, $H_1^i H^k$ is always effective, and clearly H, H_1 form also a minimum base—it will be the one used here. Before proceeding with direct applications, some preliminary propositions necessary to obtain the genera $[H_1^i H^k]$, will be proved.

12. **THEOREM.** *If V_r, V_r' are two loci of flats, having in common a curve which meets the generators of each in one point only, they can be birationally transformed into each other.*

Let S_{r+k} be the space containing them both, P an arbitrary point of V_r , G the generator through P , m the point where this generator cuts the curve common to the two varieties. Through m there goes a generator G' of V_r' . If S_k is an arbitrary k -flat in S_{r+k} , the $(k + 1)$ -flat determined by S_k and the point P , cuts G' in a single point P' . Conversely, given

¹⁶ The proof can be given in various ways. For example by generalizing Severi's for $r = 2$. See: *Sulle corrispondenze fra i punti di due curve e sopra certi classi di superficie*, Memorie della R. Acc. delle Sc. di Torino, 1903, p. 22. Certain classical formulas of his and Segre's regarding scrolls, will also be found there.

¹⁷ Unless V_r is a *hypercone*, that is such that its generators all go through the same S_{r-1} , called the *axis* of the hypercone. We will assume that this is not the case, though all results obtained hold also for a hypercone, when only variable intersections are considered.

P' on V_r' , the point P on V_r , is determined in one and only one way. Clearly P, P' are in 1-1 correspondence, and this correspondence defines a birational transformation, which proves the theorem.

Corollaries. I. *If the two varieties have in common a d -dimensional manifold M , having an S_{d-1} for complete intersection with an arbitrary generator of either, V_r and V_r' can also be birationally transformed into each other.*

For M is clearly a locus of flats—the flats S_{d-1} —and a plane section of it, is a curve satisfying all conditions of the preceding theorem.

II. *If the conditions of the theorem, or corollary I, are satisfied then $[V_r] = [V_r']$.*

For two varieties which can be birationally transformed into each other, have the same arithmetic genus.¹⁸

13. THEOREM. V_r being the same locus of flats as before $[V_r] = (-1)^{r-1}$

p . This proposition was proved by Cayley¹⁹ for the case $r = 2$, and for $r = 3$ it follows readily from a formula of Severi.²⁰

Any V_r locus of $(r - 1)$ -flats can be considered as the hyperplane section of a V_{r+1} locus of r -flats. Its hyperplane section H is itself the locus of the $(r - 2)$ -flats HG , which form a pencil of genus p . It is therefore sufficient to prove that $H = (-1)^{r-2} \cdot p$. This will be done by induction. For $r = 2$, H is the plane section of a ruled surface and the proposition is trivial. We will assume that $r > 2$, that the theorem to be proved is true for a locus of flats of less than $r - 2$ dimensions, and prove its correctness for H .

We may suppose that V_r is immersed in an S_{r+1} , and this without any loss of generality, since we can always project it into such a space and reason upon the projection. Let Φ be an r -dimensional quadratic variety through a fixed generator G_0 of V_r , and call A its residual intersection with V_r . We say that A is in one-one relation with H , arbitrary hyperplane section of V_r . For let π be an arbitrary but fixed plane of G_0 . When $r = 3$, π coincides with G_0 . If V_r is not a hypercone, the generator G through any point P of the variety intersects G_0 into an S_{r-3} , which meets π in a single point m . This point is in Φ , hence the line mP cuts the quadratic variety in one point only, other than m , say P' , a point which is in A . To P corresponds one point P' . Conversely, given P' arbitrary in A , m is determined as the point where G cuts π , and P is determined as the point where the line mP' cuts the hyperplane which intersects V_r into H . If V_r is a hypercone, we can take as point m , any

¹⁸ Severi, *Fundamenti*. . . , p. 83.

¹⁹ "On the deficiency of certain surfaces," *Math. Ann.*, Vol. III, 1871, p. 526.

²⁰ *Fundamenti*. . . , p. 81.

point where Φ cuts the axis, and proceed as in the general case. In all cases P, P' , and therefore also A and H , are in one-one correspondence.

It follows from this that $[A] = [H]$.

On the other hand, HH_1 cuts a generator HG of H , in a manifold HH_1G , which is an $(r-3)$ -flat, since H_1 is the residual intersection of V_r with an S_r through a generator. Furthermore H_1 is, like H , a locus of $(r-2)$ -flats, its generators being manifolds H_1G , each of which has in common with H_1 the same $r-3$ flat HH_1G , which has just been considered.

$$\therefore [H] = [H_1] \quad (\text{Coroll. II, § 12}).$$

Also $A = H + H_1$, hence, by formula (1), since $r > 2$,

$$[A] = [H + H_1 + HH_1] = 2[H] + [HH_1] = [H];$$

$$\therefore [H] = -[HH_1].$$

But HH_1 is the hyperplane section of H_1 , and therefore a locus of $(r-3)$ -flats of a pencil of genus p , and by assumption the proposition is true for such a variety.

$$\therefore [HH_1] = (-1)^{r-3} \cdot p, \quad [H] = (-1)^{r-2}p,$$

which proves the theorem.

14. We are now in position to obtain the genera $[H^i H_1^k]$, which are necessary to solve the fundamental problem for V_r . The manifold $H^{i+1} H_1^k$ is an $r-i-k-1$ space, if $i+k < r$. Hence

$$[H^i H_1^k] = (-1)^{r-i-k-1} \cdot p; \quad (i+k < r).$$

Let μ be the order of V_r . Then $[H^r] = \mu$. To obtain $[H^{r-k} H_1^k]$, remark that applying formula (16), we have, since $[H^{r-k} H_1^{k-1} G] = 1$,

$$[H^{r-k} H_1^{k-1} (H_1 + G)] = [H^{r-k+1} H_1^{k-1}] = 1 + [H^{r-k} H_1^k];$$

$$\therefore [H^{r-k} H_1^k] = [H^{r-k+1} H_1^{k-1} - 1].$$

Applying this last formula for all values of k , from 0 to k , and adding the results, we obtain

$$[H^{r-k} H_1^k] = \mu - k.$$

15. Let now C be any hypersurface in V_r . We have

$$C = \lambda H + \lambda_1 H_1$$

where λ, λ_1 , are certain integers. Applying (14) we obtain

$$\{C\} = [(1+H)^\lambda (1+H_1)^{\lambda_1} - 1] - \binom{\lambda + \lambda_1 - 1}{r-1}.$$

Expanding, and replacing the genera by their values as derived in § 14, we have, after some familiar transformations of combinatory analysis,

$$\{C\} = \binom{\lambda + \lambda_1 - 1}{r - 1} \left(p + \mu \frac{\lambda + \lambda_1}{r} - \lambda_1 - 1 \right) + (-1)^r p.$$

Let a be the order of C , α that of its intersection CG with G . Then, by (16),

$$\alpha = [CH^{r-2}G] = [H^{r-2}(H - H_1)(\lambda H + \lambda_1 H_1)] = \lambda + \lambda_1,$$

$$a = [CH^{r-1}] = [(\lambda H + \lambda_1 H_1)H^{r-1}] = \mu\lambda + \lambda_1(\mu - 1) = \alpha\mu - \lambda_1.$$

Substituting in the formula obtained, we have finally

$$\{C\} = \binom{\alpha - 1}{r - 1} \left(p - 1 + \frac{\mu\alpha}{r} + a - \alpha\mu \right) + (-1)^r p. \quad (17)$$

For $r = 2$, this formula becomes a well-known one of Segre,²¹ namely

$$\{C\} = [C] = \alpha p - \mu \binom{\alpha}{2} + (a - 1)(\alpha - 1).$$

16. Let $M = C_1 C_2 \cdots C_h$, with $C_i = \lambda^i H + \lambda_1^i H_1$. To derive the genus of M , we might merely apply formula (15). However, it will be done more readily in the following way: A locus V_{r-h+1} of $(r-h)$ -flats, which are in one-one correspondence with the generators of V_r , is of the same nature as V_r , and may be assumed to lie in the same S_{r+h} . Projecting the intersection of any generator of V_r with M , on the corresponding generator of V_{r-h+1} , a hypersurface M' of the latter variety is obtained. The orders of M and M' , or of MG and $M'G'$, are the same— G' being a generator of V_{r-h+1} . Hence, it is sufficient to obtain the order a of M , and α of MG , and then to apply (17) on V_{r-h+1} , since $[M] = [M']$. Let a_i , α_i , be the order of C_i and $C_i G$, respectively. We have as previously,

$$\alpha_i = \lambda^i + \lambda_1^i, \quad a_i = \mu\alpha_i - \lambda_1^i.$$

Hence, by (16),

$$a = \left[\prod_{i=1}^h (\lambda^i H + \lambda_1^i H_1) H^{r-h} \right] = \alpha \left(\mu + \sum_{i=1}^h \frac{a_i - \mu\alpha_i}{\alpha_i} \right),$$

$$\alpha = \left[\prod_{i=1}^h (\lambda^i H + \lambda_1^i H_1) H^{r-h-1} G \right] = \prod_{i=1}^h \alpha_i.$$

Substituting in (17), we have

$$\{M\} = \binom{\alpha - 1}{r - h} \left(a - \mu\alpha + p - 1 + \frac{\mu\alpha}{r - h + 1} \right) + (-1)^{r-h+1} \cdot p.$$

²¹ "Courbes et surfaces réglées algébriques," Math. Ann., Vol. 34, 1890, p. 3.

For $h = r$ we have either from this last formula, or from (16),

$$[C_1 C_2 \cdots C_r] = \left[\prod_{i=1}^r (\lambda^i H + \lambda_1^i H_1) \right] = \prod_{i=1}^r \alpha_i \left(\sum_{i=1}^r \frac{a_i}{\alpha_i} - (r-1)\mu \right),$$

which for $r = 2$, reduces to a well known relation on scrolls.

17. **Complete intersections in S_r .** In the case of an S_r , $\rho = 1$, and the minimum base is formed by an $(r-1)$ -flat, say C . If m is the order of a V_r in S , we have $V_r = mC$. Also $[C^k] = 0$, if $k < r$, and $[C^r] = 1$. Hence if M is the complete intersection of h hypersurfaces of order $m_1, m_2, \dots, m_h$, we obtain, by applying (15), when $h < r$ and with $m = \Sigma m_i$,

$$\begin{aligned} \{M\} = [M] &= \left[\prod_{i=1}^h (1 + C)^{m_i} - 1 \right] - \binom{m-1}{r-1} + \sum_{i=1}^{i=h} \binom{m-m_i-1}{r-1} \\ &\quad + \cdots + (-1)^h \sum_{i=1}^{i=h} \binom{m_i-1}{r-1} \\ &= \binom{m-1}{r} - \sum_{i=1}^{i=h} \binom{m-m_i-1}{r} \\ &\quad + \cdots + (-1)^{h-1} \sum_{i=1}^{i=h} \binom{m_i-1}{r}. \end{aligned}$$

The value of $[M]$ may be also obtained as follows: If $v(l)$ is the postulation of $[M]$, then

$$v(l) = \binom{l+r}{r} - \sum_i \binom{l-m_i+r}{r} + \sum_{i,j,i \neq j} \binom{l-m_i-m_j+r}{r} - \dots$$

Remembering that $[M] = (-1)^{r-h} \cdot (v(0) - 1)$, we can from $v(l)$ obtain $[M]$ and, in fact, the same expression as the above is found. Conversely Severi's general expression²³ for $v(l)$ together with the formula derived here for $[M]$, can be utilized to obtain Bertini's result.

²² Bertini, *Introduzione alla geometria proiettiva degli iperspazi*, p. 263.

²³ *Fundamenti*. . . , p. 41.